

# Semantics and Type Soundness Proof of a Core Fragment of Fortress with Hidden Type Variables

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## Notation

- We use  $C$  as a meta-variable for object and trait names, i.e.  $O$  and  $T$ .
- We use  $\Gamma$  as a meta-variable for term variable environments.
- We use  $\Delta$  as a meta-variable for type variable environments.
- We use  $\vec{\tau}$  as short-hand notation for  $\tau_1 \dots \tau_n$ . We write the empty sequence as  $\cdot$ . We denote the concatenation of two sequences by juxtaposition, such as  $\vec{\tau} \vec{\sigma}$ . We extend this notation to other entities beyond types.
- We define  $name(md)$  to be the name of method  $md$ .
- We define the function  $min-path_{p\Delta}$  such that  $min-path_{p\Delta}(\tau, \vec{\sigma})$  is a minimal path,  $\tau \vec{\rho}$ , where:
  - $p; \Delta \vdash \tau \vec{\rho}$  path-ok,
  - every  $\pi$  in  $\vec{\sigma}$  is also in  $\vec{\rho}$ .

## Restrictions

- We assume all type variables are unique.
- We require the type `Object` to be the root of the type hierarchy created by any valid program.
- Hidden type variables cannot occur in the type of a field.
- We require no cycles in the bounds relation. That is, a type parameter declaration  $\alpha \triangleleft \beta$ , along with the declaration  $\beta \triangleleft \alpha$  is forbidden. However, recursive and mutually recursive bounds, such as  $\alpha \triangleleft T[\alpha]$  and  $\alpha \triangleleft T[\beta]$  along with  $\beta \triangleleft T'[\alpha]$  are allowed.
- Allowing a trait to extend other instantiations of itself raises the possibility of a cyclic type hierarchy. Consider:

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trait T[ $\alpha \triangleleft$  Object]  $\triangleleft$  T[ $\beta$ ] | { $\beta \triangleleft$  Object} end
```

Then we have  $T[T[\text{Object}]] \triangleleft T[\text{Object}] \triangleleft T[T[\text{Object}]]$ .

We must prevent cyclic type hierarchies in order to keep the semantics of method inheritance and overriding well defined. It is not particularly relevant to our system how such cycles are prevented, but they must be prevented in some way. Obviously, we could simply restrict cyclic hierarchies by fiat, but such a restriction is not useful unless there is an effective algorithm for enforcing it.

We say that a trait  $T$  is *self-extending* if one of its supertype  $N_i$  is an instantiation of  $T$ ; we also say that  $N_i$  is a *self-supertype* of  $T$ . Then we place the following restrictions on a trait definition:

- (R1) There are no cycles except through self-extension.
- (R2) A trait definition has at most one self-supertype.

(R3) Each type parameter of a trait either bounds or is bound by the corresponding type argument in the self-supertype.

These restrictions permit traits to have invariant, covariant and contravariant type parameters.

- Method overloading is forbidden. It is not allowed for a trait or object provide (either define or inherit) two method with the same name.

Syntax:

$p$	$::= \vec{d} e$	program
$d$	$::= td$   $od$	
$td$	$::= \mathbf{trait} T[\overline{\alpha \triangleleft K}] \triangleleft \overline{N} \mid \{\overline{\alpha \triangleleft K}\} \overline{md} \mathbf{end}$	trait definition
$od$	$::= \mathbf{object} O[\overline{\alpha \triangleleft K}](\overline{x:\vec{\tau}}) \triangleleft \overline{N} \mid \{\overline{\alpha \triangleleft K}\} \overline{md} \mathbf{end}$	object definition
$md$	$::= m[\overline{\alpha \triangleleft K}](\overline{x:\vec{\tau}}):\tau = e$	method definition
$e$	$::= x$   $\mathbf{self}$   $O[\overline{\vec{\tau}}](\overline{e})$   $e.x$   $e \mathbf{as} (\overline{\vec{\tau}}, \overline{\vec{\tau}}).m[\overline{\vec{\tau}}](\overline{e})$   $\mathbf{typecase} x = e \mathbf{in} \overline{\vec{\tau}} \Rightarrow \overline{e} \mathbf{else} e \mathbf{end}$   $e \mathbf{as} \tau$	variable reference self reference object creation field access method invocation type-dependent operation
$\tau, \sigma, \rho, \pi$	$::= O[\overline{\vec{\tau}}]$   $K$	type ascription object type
$K, L$	$::= \alpha$   $N$	type variable
$M, N$	$::= T[\overline{\vec{\tau}}]$   $\mathbf{Object}$	trait type Object type

Expression typing:  $p; \Delta; \Gamma \vdash e : \tau$

[T-VAR]  $p; \Delta; \Gamma \vdash x : \Gamma(x)$

[T-SELF]  $p; \Delta; \Gamma \vdash \mathbf{self} : \Gamma(\mathbf{self})$

[T-OBJECT]

$$\frac{\begin{array}{l} \text{object } O[\overrightarrow{\alpha \triangleleft \_}] (\overrightarrow{\_} : \overrightarrow{\sigma}) \_ \in p \\ p; \Delta \vdash O[\overrightarrow{\tau}] \text{ok} \quad p; \Delta; \Gamma \vdash \overrightarrow{e} : \overrightarrow{\rho} \\ p; \Delta \vdash \overrightarrow{\rho} <: [\overrightarrow{\tau} / \overrightarrow{\alpha}] \overrightarrow{\sigma} \end{array}}{p; \Delta; \Gamma \vdash O[\overrightarrow{\tau}](\overrightarrow{e}) : O[\overrightarrow{\tau}]}$$

[T-FIELD]

$$\frac{\begin{array}{l} \text{object } O[\overrightarrow{\alpha \triangleleft \_}] (\overrightarrow{x} : \overrightarrow{\sigma}) \_ \in p \\ p; \Delta; \Gamma \vdash e : O[\overrightarrow{\tau}] \end{array}}{p; \Delta; \Gamma \vdash e.x_i : [\overrightarrow{\tau} / \overrightarrow{\alpha}] \sigma_i}$$

[T-METHOD]

$$\frac{\begin{array}{l} p; \Delta; \Gamma \vdash e : \tau \quad p; \Delta \vdash \tau \overrightarrow{\sigma} \text{ path-ok} \\ p; \Delta \vdash \overrightarrow{\rho} \text{ ok} \\ \text{method}_p(m, \tau \overrightarrow{\sigma}, \overrightarrow{\rho}) = \{(m[\overrightarrow{\alpha \triangleleft K}](\overrightarrow{\_} : \overrightarrow{\nu}) : \kappa \_, \overrightarrow{\omega}_1 <: \overrightarrow{\omega}_2)\} \\ p; \Delta; \Gamma \vdash \overrightarrow{e} : \overrightarrow{\nu}' \quad p; \Delta \vdash \overrightarrow{\nu}' <: [\overrightarrow{\pi} / \overrightarrow{\alpha}] \overrightarrow{\nu} \\ p; \Delta \vdash \overrightarrow{\pi} \text{ ok} \quad p; \Delta \vdash \overrightarrow{\pi} <: [\overrightarrow{\pi} / \overrightarrow{\alpha}] \overrightarrow{K} \\ p; \Delta \vdash \overrightarrow{\omega}_1 <: \overrightarrow{\omega}_2 \end{array}}{p; \Delta; \Gamma \vdash e \text{ as } (\tau \overrightarrow{\sigma}, \overrightarrow{\rho}).m[\overrightarrow{\pi}](\overrightarrow{e}) : [\overrightarrow{\pi} / \overrightarrow{\alpha}] \kappa}$$

[T-TYPECASE]

$$\frac{\begin{array}{l} p; \Delta; \Gamma \vdash e : \rho \\ p; \Delta; \Gamma x : \tau_i \vdash e_i : \pi_i \quad p; \Delta \vdash \pi_i <: \sigma \quad 1 \leq i \leq |\overrightarrow{\tau}| \\ p; \Delta; \Gamma x : \rho \vdash e' : \nu \quad p; \Delta \vdash \nu <: \sigma \end{array}}{p; \Delta; \Gamma \vdash \text{typecase } x = e \text{ in } \overrightarrow{\tau} \Rightarrow \overrightarrow{e} \text{ else } e' \text{ end} : \sigma}$$

[T-ASCRPTION]

$$\frac{p; \Delta; \Gamma \vdash e : \sigma \quad p; \Delta \vdash \sigma <: \tau}{p; \Delta; \Gamma \vdash e \text{ as } \tau : \tau}$$

Method lookup:  $\boxed{\text{method}_p(m, \overrightarrow{\tau}, \overrightarrow{\tau}) = \{(md_1, \overrightarrow{\tau} <: \overrightarrow{\tau}), \dots, (md_n, \overrightarrow{\tau} <: \overrightarrow{\tau})\}}$

$\text{method}_p(m, \cdot, \overrightarrow{\tau}) = \emptyset$

$\text{method}_p(m, \text{Object}, \overrightarrow{\tau}) = \emptyset$

$\text{method}_p(m, \alpha \overrightarrow{\sigma}, \overrightarrow{\rho}) = \text{method}_p(m, \overrightarrow{\sigma}, \overrightarrow{\rho})$

$$\left\{ \begin{array}{l} \text{method}_p(m, C[\overrightarrow{\tau}] \overrightarrow{\sigma}, \overrightarrow{\rho}) = \\ \left\{ \begin{array}{l} \{([\overrightarrow{\tau} / \overrightarrow{\alpha}][\overrightarrow{\rho} / \overrightarrow{\beta}] md, [\overrightarrow{\tau} / \overrightarrow{\alpha}][\overrightarrow{\rho} / \overrightarrow{\beta}] \Delta)\} \quad \text{if } - C[\overrightarrow{\alpha \triangleleft K}] \_ - \{ \overrightarrow{\beta \triangleleft L} \} \overrightarrow{md} \_ \in p \\ \text{and } md \in \{\overrightarrow{md}\} \\ \text{and } \text{name}(md) = m \\ \text{and } \Delta = \overrightarrow{\alpha} <: \overrightarrow{K} \overrightarrow{\beta} <: \overrightarrow{L} \end{array} \right. \\ \text{method}_p(m, \overrightarrow{\sigma}, \overrightarrow{\rho}) \quad \text{if } - C \_ \overrightarrow{md} \_ \in p \\ \text{and } m \notin \{\text{name}(md)\} \end{array} \right.$$

Extension:  $\boxed{p; \Delta \vdash \tau <:: \tau}$

$$[\text{E-VAR}] \quad \frac{\tau \in \Delta(\alpha)}{p; \Delta \vdash \alpha <:: \tau}$$

$$[\text{E-EXT}] \quad \frac{\begin{array}{l} - C[\overrightarrow{\alpha} \triangleleft \overrightarrow{K}] \triangleleft \overrightarrow{M} T[\overrightarrow{\rho}] \overrightarrow{N} \mid \{\overrightarrow{\beta} \triangleleft \overrightarrow{L}\} - \in p \\ p; \Delta \vdash \overrightarrow{\tau} \text{ ok} \quad p; \Delta \vdash \overrightarrow{\tau} <: [\overrightarrow{\tau}/\overrightarrow{\alpha}][\overrightarrow{\sigma}/\overrightarrow{\beta}]\overrightarrow{K} \\ p; \Delta \vdash \overrightarrow{\sigma} \text{ ok} \quad p; \Delta \vdash \overrightarrow{\sigma} <: [\overrightarrow{\tau}/\overrightarrow{\alpha}][\overrightarrow{\sigma}/\overrightarrow{\beta}]\overrightarrow{L} \end{array}}{p; \Delta \vdash C[\overrightarrow{\tau}] <:: [\overrightarrow{\tau}/\overrightarrow{\alpha}][\overrightarrow{\sigma}/\overrightarrow{\beta}]T[\overrightarrow{\rho}]}$$

Subtyping:  $\boxed{p; \Delta \vdash \tau <: \tau}$

$$[\text{S-REFL}] \quad p; \Delta \vdash \tau <: \tau$$

$$[\text{S-TRANS}] \quad \frac{p; \Delta \vdash \tau_1 <: \tau_2 \quad p; \Delta \vdash \tau_2 <: \tau_3}{p; \Delta \vdash \tau_1 <: \tau_3}$$

$$[\text{S-EXT}] \quad \frac{p; \Delta \vdash \tau_1 <:: \tau_2}{p; \Delta \vdash \tau_1 <: \tau_2}$$

Program typing:  $\boxed{\vdash p : \tau}$

$$[\text{T-PROGRAM}] \quad \frac{p = \overrightarrow{d} \ e \quad p \vdash \overrightarrow{d} \text{ ok} \quad p; \emptyset; \emptyset \vdash e : \tau}{\vdash p : \tau}$$

Definition typing:  $\boxed{p \vdash d \text{ ok}}$

$$[\text{T-TRAITDEF}] \quad \frac{\begin{array}{l} \Delta = \overrightarrow{\alpha} <: \overrightarrow{K} \ \overrightarrow{\beta} <: \overrightarrow{L} \\ p; \Delta \vdash \overrightarrow{K} \text{ ok} \quad p; \Delta \vdash \overrightarrow{N} \text{ ok} \quad p; \Delta \vdash \overrightarrow{L} \text{ ok} \\ p; \Delta; \text{self} : T[\overrightarrow{\alpha}]; T[\overrightarrow{\alpha}][\overrightarrow{\beta}] \vdash \overrightarrow{md} \text{ ok} \end{array}}{p \vdash \text{trait } T[\overrightarrow{\alpha} \triangleleft \overrightarrow{K}] \triangleleft \overrightarrow{N} \mid \{\overrightarrow{\beta} \triangleleft \overrightarrow{L}\} \overrightarrow{md} \text{ end ok}}$$

$$[\text{T-OBJECTDEF}] \quad \frac{\begin{array}{l} \Delta = \overrightarrow{\alpha} <: \overrightarrow{K} \\ \Delta' = \overrightarrow{\alpha} <: \overrightarrow{K} \ \overrightarrow{\beta} <: \overrightarrow{L} \\ p; \Delta' \vdash \overrightarrow{K} \text{ ok} \quad p; \Delta \vdash \overrightarrow{\tau} \text{ ok} \\ p; \Delta' \vdash \overrightarrow{N} \text{ ok} \quad p; \Delta' \vdash \overrightarrow{L} \text{ ok} \\ p; \Delta'; \text{self} : O[\overrightarrow{\alpha}] \overrightarrow{x} : \overrightarrow{\tau}; O[\overrightarrow{\alpha}][\overrightarrow{\beta}] \vdash \overrightarrow{md} \text{ ok} \end{array}}{p \vdash \text{object } O[\overrightarrow{\alpha} \triangleleft \overrightarrow{K}](\overrightarrow{x} : \overrightarrow{\tau}) \triangleleft \overrightarrow{N} \mid \{\overrightarrow{\beta} \triangleleft \overrightarrow{L}\} \overrightarrow{md} \text{ end ok}}$$

Method typing:  $\boxed{p; \Delta; \Gamma; C[\vec{\alpha}][\vec{\alpha}] \vdash md \text{ ok}}$

[T-METHODDEF]

$$\frac{\begin{array}{c} \Delta' = \Delta \overline{\alpha} <: \vec{K} \\ p; \Delta; C[\vec{\beta}][\vec{\gamma}] \vdash \text{override}(m[\overline{\alpha} \triangleleft \vec{K}](\overline{x} : \vec{\sigma}) : \rho = e) \\ p; \Delta' \vdash \vec{K} \text{ ok} \quad p; \Delta' \vdash \vec{\sigma} \text{ ok} \quad p; \Delta' \vdash \rho \text{ ok} \\ p; \Delta'; \Gamma \overline{x} : \vec{\sigma} \vdash e : \pi \quad p; \Delta' \vdash \pi <: \rho \end{array}}{p; \Delta; \Gamma; C[\vec{\beta}][\vec{\gamma}] \vdash m[\overline{\alpha} \triangleleft \vec{K}](\overline{x} : \vec{\sigma}) : \rho = e \text{ ok}}$$

Method overriding:  $\boxed{p; \Delta; C[\vec{\alpha}][\vec{\alpha}] \vdash \text{override}(md)}$

[OVERRIDE]

$$\frac{\begin{array}{c} p; \Delta \vdash C[\vec{\beta}][\vec{\rho}] \text{ path-ok} \\ \text{method}_p(m, \vec{\rho}, \vec{\gamma}) = \{(m[\overline{\alpha'} \triangleleft \vec{K}'](\overline{-} : \vec{\pi}) : \nu \_, \overline{\kappa}_1 <: \overline{\kappa}_2)\} \\ \vec{K} = [\vec{\alpha} / \overline{\alpha'}] \vec{K}' \\ \vec{\tau} = [\vec{\alpha} / \overline{\alpha'}] \vec{\pi} \quad p; \Delta \vdash \sigma <: [\vec{\alpha} / \overline{\alpha'}] \nu \\ p; \Delta \vdash \overline{\kappa}_1 <: \overline{\kappa}_2 \end{array}}{p; \Delta; C[\vec{\beta}][\vec{\gamma}] \vdash \text{override}(m[\overline{\alpha} \triangleleft \vec{K}](\overline{x} : \vec{\tau}) : \sigma = e)}$$

[NONOVERRIDE]

$$\frac{\begin{array}{c} p; \Delta \vdash C[\vec{\beta}][\vec{\rho}] \text{ path-ok} \\ \text{method}_p(m, \vec{\rho}, \vec{\gamma}) = \emptyset \end{array}}{p; \Delta; C[\vec{\beta}][\vec{\gamma}] \vdash \text{override}(m[\overline{\alpha} \triangleleft \vec{K}](\overline{x} : \vec{\tau}) : \sigma = e)}$$

Well-formed types:  $\boxed{p; \Delta \vdash \tau \text{ ok}}$

[W-OBJ]  $p; \Delta \vdash \text{Object ok}$

[W-VAR]  $\frac{\alpha \in \text{dom}(\Delta)}{p; \Delta \vdash \alpha \text{ ok}}$

[W-TAPP]  $\frac{\begin{array}{c} \_ \in p \\ p; \Delta \vdash \vec{\tau} \text{ ok} \quad p; \Delta \vdash \vec{\sigma} \text{ ok} \\ p; \Delta \vdash \vec{\tau} <: [\vec{\tau} / \overline{\alpha}][\vec{\sigma} / \vec{\beta}] \vec{K} \\ p; \Delta \vdash \vec{\sigma} <: [\vec{\tau} / \overline{\alpha}][\vec{\sigma} / \vec{\beta}] \vec{L} \end{array}}{p; \Delta \vdash C[\vec{\tau}] \text{ ok}}$

Well-formed paths:  $\boxed{p; \Delta \vdash \vec{\tau} \text{ path-ok}}$

[PATH]  $\frac{\begin{array}{c} p; \Delta \vdash \tau_1 <: \tau_2 \\ \dots \\ p; \Delta \vdash \tau_{n-1} <: \tau_n \end{array}}{p; \Delta \vdash \tau_1 \dots \tau_n \text{ path-ok}}$

Evaluation contexts and redexes

$$\begin{array}{l}
v ::= O[\vec{\tau}](\vec{v}) \\
E ::= \square \\
\quad | O[\vec{\tau}](\vec{v} E \vec{e}) \\
\quad | E.x \\
\quad | E \text{ as } (\vec{\tau}, \vec{\tau}).m[\vec{\tau}](\vec{e}) \\
\quad | v \text{ as } (\vec{\tau}, \vec{\tau}).m[\vec{\tau}](\vec{v} E \vec{e}) \\
\quad | \text{typecase } x = E \text{ in } \vec{\tau} \Rightarrow \vec{e} \text{ else } e \text{ end} \\
R ::= v.x \\
\quad | v \text{ as } (\vec{\tau}, \vec{\tau}).m[\vec{\tau}](\vec{v}) \\
\quad | \text{typecase } x = v \text{ in } \vec{\tau} \Rightarrow \vec{e} \text{ else } e \text{ end} \\
\quad | e \text{ as } \tau
\end{array}$$

Evaluation rules:  $\boxed{p \vdash E[R] \longrightarrow E[e]}$

[R-FIELD]

$$\frac{\text{object } O \_ (x: \_)\_ \in p}{p \vdash E[O[\vec{\tau}](\vec{v}).x_i] \longrightarrow E[v_i]}$$

[R-METHOD]

$$\frac{\text{object } O \_ (x: \_)\_ \text{ end} \in p \quad \text{method}_p(m, \vec{\sigma}, \vec{\rho}) = \{(m[\vec{\alpha} \triangleleft \vec{\_}](x': \_)\_ = e, \_)\}}{p \vdash E[O[\vec{\tau}](\vec{v}) \text{ as } (\vec{\sigma}, \vec{\rho}).m[\vec{\pi}](\vec{v}')] \longrightarrow E[\vec{v}/\vec{x}][O[\vec{\tau}](\vec{v})/\text{self}][\vec{v}'/\vec{x}'][\pi/\alpha]e]}$$

[R-TYPECASE]

$$\frac{p; \emptyset \not\vdash O[\vec{\tau}] <: \sigma_i \quad 1 \leq i < |\vec{\sigma}| \quad p; \emptyset \vdash O[\vec{\tau}] <: \rho}{p \vdash E[\text{typecase } x = O[\vec{\tau}](\vec{v}) \text{ in } \vec{\sigma} \Rightarrow e_1 \rho \Rightarrow e_2 \vec{\pi} \Rightarrow e_3 \text{ else } e_4 \text{ end}] \longrightarrow E[[O[\vec{\tau}](\vec{v})/x]e_2]}$$

[R-TYPECASEELSE]

$$\frac{p; \emptyset \not\vdash O[\vec{\tau}] <: \sigma_i \quad 1 \leq i \leq |\vec{\sigma}|}{p \vdash E[\text{typecase } x = O[\vec{\tau}](\vec{v}) \text{ in } \vec{\sigma} \Rightarrow e_1 \text{ else } e_2 \text{ end}] \longrightarrow E[[O[\vec{\tau}](\vec{v})/x]e_2]}$$

[R-ASCRPTION]

$$\frac{}{p \vdash E[e \text{ as } \tau] \longrightarrow E[e]}$$

Path update:  $\boxed{\text{path}_{p\Delta\Gamma}(e) = e}$

$$\begin{array}{l}
\text{path}_{p\Delta\Gamma}(e) = \\
\left\{ \begin{array}{l}
O[\vec{\tau}](\vec{e}') \quad \text{if } e = O[\vec{\tau}](\vec{e}') \\
\quad \text{and } \vec{e}' = \text{path}_{p\Delta\Gamma}(\vec{e}) \\
e''.x \quad \text{if } e = e'.x \\
\quad \text{and } e'' = \text{path}_{p\Delta\Gamma}(e') \\
e'' \text{ as } (\vec{\nu}, \vec{\sigma}).m[\vec{\tau}](\vec{e}') \quad \text{if } e = e' \text{ as } (\vec{\tau}, \vec{\sigma}).m[\vec{\rho}](\vec{e}') \\
\quad \text{and } e'' = \text{path}_{p\Delta\Gamma}(e') \\
\quad \text{and } p; \Delta; \Gamma \vdash e'' : \pi \\
\quad \text{and } \vec{\nu} = \text{min-path}_{p\Delta}(\pi, \vec{\tau}) \\
\quad \text{and } \vec{e}' = \text{path}_{p\Delta\Gamma}(\vec{e}) \\
\text{typecase } x = e'_1 \text{ in } \vec{\sigma} \Rightarrow e'_2 \text{ else } e'_3 \text{ end} \quad \text{if } e = \text{typecase } x = e_1 \text{ in } \vec{\sigma} \Rightarrow e_2 \text{ else } e_3 \text{ end} \\
\quad \text{and } e'_1 = \text{path}_{p\Delta\Gamma}(e_1) \\
\quad \text{and } e'_2 = \text{path}_{p\Delta\Gamma}(e_2) \\
\quad \text{and } e'_3 = \text{path}_{p\Delta\Gamma}(e_3) \\
e'' \text{ as } \tau \quad \text{if } e = e' \text{ as } \tau \\
\quad \text{and } e'' = \text{path}_{p\Delta\Gamma}(e') \\
e \quad \text{otherwise}
\end{array} \right.
\end{array}$$

### Type Soundness Proof

**Theorem 1** (Progress). *If  $p; \emptyset; \emptyset \vdash e_1 : \tau$  then either  $e_1$  is a value or  $p \vdash e_1 \longrightarrow e_2$ .*

*Proof.* The proof is by case analysis on the current redex in  $e_1$  (in the case that  $e_1$  is not a value).

Case  $O[\vec{\tau}](\vec{v}).x_i$ : By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash O[\vec{\tau}](\vec{v}).x_i : [\vec{\tau}/\vec{\alpha}]\sigma_i$$

where:

$$\text{object } O[\vec{\alpha} \triangleleft \_](\vec{x}:\vec{\sigma}) \_ \in p.$$

Therefore [R-FIELD] can be applied.

Case  $O[\vec{\tau}](\vec{v}) \text{ as } (\vec{\sigma}, \vec{\rho}).m[\vec{\pi}](\vec{v}')$ : By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash O[\vec{\tau}](\vec{v}) \text{ as } (\vec{\sigma}, \vec{\rho}).m[\vec{\pi}](\vec{v}') : [\vec{\pi}/\vec{\alpha}]\kappa$$

where:

$$\text{method}_p(m, \vec{\sigma}, \vec{\rho}) = \{(m[\vec{\alpha} \triangleleft \vec{K}](\vec{x}:\vec{\nu}):\kappa = e, \_)\}.$$

Therefore [R-METHOD] can be applied.

Case  $\text{typecase } x = v \text{ in } \vec{\tau} \Rightarrow \vec{e} \text{ else } e \text{ end}$ : One of [R-TYPECASE] or [R-TYPECASEELSE] must apply.

Case  $e \text{ as } \tau$ : Then rule [R-ASCRPTION] applies. □

**Lemma 1** (Weakening). *Suppose  $p; \Delta \overline{\alpha} <: \vec{K} \vdash \vec{K}$  ok and  $p; \Delta \vdash \sigma$  ok.*

1. *If  $p; \Delta \vdash \tau_1 <: \tau_2$  then  $p; \Delta \overline{\alpha} <: \vec{K} \vdash \tau_1 <: \tau_2$ .*
2. *If  $p; \Delta \vdash \tau$  ok then  $p; \Delta \overline{\alpha} <: \vec{K} \vdash \tau$  ok.*
3. *If  $p; \Delta; \Gamma \vdash e : \tau$  then  $p; \Delta \overline{\alpha} <: \vec{K}; \Gamma \vdash e : \tau$  and  $p; \Delta; \Gamma x : \sigma \vdash e : \tau$ .*

*Proof.* Each of them is proved by straightforward induction on the derivation of  $p; \Delta \vdash \tau_1 <: \tau_2$ ,  $p; \Delta \vdash \tau$  ok, and  $p; \Delta; \Gamma \vdash e : \tau$ . □

**Lemma 2** (Type Substitution Preserves Well-Formedness). *If  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau$  ok and  $p; \Delta_1 \vdash \overrightarrow{\tau} <: [\overrightarrow{\tau}/\overrightarrow{\alpha}]\overrightarrow{K}$  with  $p; \Delta_1 \vdash \overrightarrow{\tau}$  ok and none of  $\overrightarrow{\alpha}$  appearing in  $\Delta_1$ , then  $p; \Delta_1 [\overrightarrow{\tau}/\overrightarrow{\alpha}]\Delta_2 \vdash [\overrightarrow{\tau}/\overrightarrow{\alpha}]\tau$  ok.*

*Proof.* By straightforward induction on the derivation of  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau$  ok.  $\square$

**Lemma 3** (Type Substitution Preserves Extension). *If  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau_1 <: \tau_2$  and  $p; \Delta_1 \vdash \overrightarrow{\sigma} <: [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\overrightarrow{K}$  with  $p; \Delta_1 \vdash \overrightarrow{\sigma}$  ok and none of  $\overrightarrow{\alpha}$  appear in  $\Delta_1$ , then  $p; \Delta_1 [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\Delta_2 \vdash [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\tau_1 <: [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\tau_2$ .*

*Proof.* By induction on the derivation of  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau_1 <: \tau_2$ .

Case [E-VAR]: Then  $\tau_1 = \alpha$  and  $\tau_2 = (\Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2)(\alpha)$ . If  $\alpha \in \text{dom}(\Delta_2)$ , then the conclusion is immediate. Otherwise,  $\alpha = \alpha_i$  and by assumption we have:

$$p; \Delta_1 \vdash \sigma_i <: [\overrightarrow{\sigma}/\overrightarrow{\alpha}]K_i.$$

Lastly, Lemma 1 gives us the desired result.

Case [E-EXT]: Follows from Lemma 2, Lemma 4, and rule [S-EXT].  $\square$

**Lemma 4** (Type Substitution Preserves Subtyping). *If  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau_1 <: \tau_2$  and  $p; \Delta_1 \vdash \overrightarrow{\sigma} <: [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\overrightarrow{K}$  with  $p; \Delta_1 \vdash \overrightarrow{\sigma}$  ok and none of  $\overrightarrow{\alpha}$  appear in  $\Delta_1$ , then  $p; \Delta_1 [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\Delta_2 \vdash [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\tau_1 <: [\overrightarrow{\sigma}/\overrightarrow{\alpha}]\tau_2$ .*

*Proof.* By induction on the derivation of  $p; \Delta_1 \overrightarrow{\alpha} <: \overrightarrow{K} \Delta_2 \vdash \tau_1 <: \tau_2$ .

Case [S-REFL]: Trivial.

Case [S-TRANS]: Follows from the induction hypothesis.

Case [S-EXT]: Follows from Lemma 3.  $\square$

**Lemma 5** (Valid Overriding). *If  $\text{method}_p(m, \overrightarrow{\tau}, \overrightarrow{\sigma}) = \{(m[\overrightarrow{\alpha} \triangleleft \overrightarrow{K}](\overrightarrow{x}:\overrightarrow{\rho}):\pi, \overrightarrow{\omega}_1 <: \overrightarrow{\omega}_2)\}$  where  $p; \Delta \vdash \overrightarrow{\omega}_1 <: \overrightarrow{\omega}_2$  and there exists  $\overrightarrow{\nu}$  such that  $p; \Delta \vdash \overrightarrow{\nu}$  path-ok and every  $\kappa$  in  $\overrightarrow{\tau}$  is also in  $\overrightarrow{\nu}$  then,  $\text{method}_p(m, \overrightarrow{\nu}, \overrightarrow{\sigma}) = \{(m[\overrightarrow{\alpha} \triangleleft \overrightarrow{K}](\overrightarrow{x}:\overrightarrow{\rho}):\pi', \overrightarrow{\omega}'_1 <: \overrightarrow{\omega}'_2)\}$  and  $p; \Delta \vdash \pi' <: \pi$  and  $p; \Delta \vdash \overrightarrow{\omega}'_1 <: \overrightarrow{\omega}'_2$ .*

*Proof.* Follows from rules [OVERRIDE] and [NONOVERRIDE].  $\square$

**Lemma 6** (Term Substitution Preserves Typing). *If  $p$  is well typed and  $p; \Delta; \Gamma \overrightarrow{x}:\overrightarrow{\tau} \vdash e : \sigma$  and  $p; \Delta; \Gamma \vdash \overrightarrow{e}' : \overrightarrow{\rho}$  and  $p; \Delta \vdash \overrightarrow{\rho} <: \overrightarrow{\tau}$  then  $p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}([\overrightarrow{e}'/\overrightarrow{x}]e) : \pi$  and  $p; \Delta \vdash \pi <: \sigma$ .*

*Proof.* By induction on the derivation of  $p; \Delta; \Gamma \overrightarrow{x}:\overrightarrow{\tau} \vdash e : \sigma$ .

Case [T-SELF]: Trivial.

Case [T-OBJECT] [T-ASCRPTION]: Easy.

Case [T-VAR]  $p; \Gamma \overrightarrow{x}:\overrightarrow{\tau}; \Delta \vdash x : \Gamma(x)$ :

If  $x \in \text{dom}(\Gamma)$  then the result is immediate. Otherwise  $x = x_i$  and  $\sigma = \tau_i$ . Then  $\pi = \rho_i$ .

Case [T-FIELD]  $\frac{\text{object } O[\overrightarrow{\alpha} \triangleleft \overrightarrow{\tau}](\overrightarrow{x}:\overrightarrow{\nu}) \_ \in p}{p; \Delta; \Gamma \overrightarrow{x}:\overrightarrow{\tau} \vdash e' : O[\overrightarrow{\kappa}]};$   
 $p; \Delta; \Gamma \overrightarrow{x}:\overrightarrow{\tau} \vdash e'.x_i : [\overrightarrow{\kappa}/\overrightarrow{\alpha}]\nu_i$

By the induction hypothesis, we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}([\overrightarrow{e}'/\overrightarrow{x}]e') : O[\overrightarrow{\kappa}]$$

Notice that:

$$\begin{aligned} & \text{path}_{p\Delta\Gamma}(\text{path}_{p\Delta\Gamma}([\overrightarrow{e}'/\overrightarrow{x}]e').x_i) \\ &= \text{path}_{p\Delta\Gamma}([\overrightarrow{e}'/\overrightarrow{x}]e'.x_i). \end{aligned}$$

Rule [T-FIELD] finishes the case.

$$\begin{array}{c}
p; \Delta; \Gamma \overline{x} : \overline{\tau} \vdash e : \sigma \quad p; \Delta \vdash \sigma \overline{\rho} \text{ path-ok} \\
p; \Delta \vdash \overline{\pi} \text{ ok} \\
method_p(m, \sigma \overline{\rho}, \overline{\pi}) = \{(m[\overline{\alpha} \triangleleft \overline{K}] (\overline{-} : \overline{\kappa}) : \omega \_ , \overline{\theta}_1 < : \overline{\theta}_2)\} \\
p; \Delta; \Gamma \overline{x} : \overline{\tau} \vdash e' : \overline{\kappa}' \quad p; \Delta \vdash \overline{\kappa}' < : [\overline{\nu} / \overline{\alpha}] \overline{\kappa} \\
p; \Delta \vdash \overline{\nu} \text{ ok} \quad p; \Delta \vdash \overline{\nu} < : [\overline{\nu} / \overline{\alpha}] \overline{K} \\
p; \Delta \vdash \overline{\theta}_1 < : \overline{\theta}_2
\end{array}$$

Case [T-METHOD]  $\frac{\quad}{p; \Delta; \Gamma \overline{x} : \overline{\tau} \vdash e \text{ as } (\sigma \overline{\rho}, \overline{\pi}).m[\overline{\nu}](e') : [\overline{\nu} / \overline{\alpha}] \omega}$ :

By the induction hypothesis we have:

$$\begin{array}{c}
p; \Delta; \Gamma \vdash path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) : \sigma' \\
p; \Delta \vdash \sigma' < : \sigma \\
p; \Delta; \Gamma \vdash path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e') : \overline{\kappa}'' \\
p; \Delta \vdash \overline{\kappa}'' < : \overline{\kappa}'.
\end{array}$$

By [S-TRANS] we have:

$$p; \Delta \vdash \overline{\kappa}'' < : [\overline{\nu} / \overline{\alpha}] \overline{\kappa}.$$

Notice that:

$$\begin{aligned}
& path_{p\Delta\Gamma}(path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ as } (\sigma, \overline{\rho}, \overline{\pi}).m[\overline{\nu}](path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e'))) \\
&= path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ as } (\overline{\rho}', \overline{\pi}).m[\overline{\nu}](path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e'))
\end{aligned}$$

where:

$$min\text{-}path_{p\Delta}(\sigma', \sigma \overline{\rho}) = \overline{\rho}'.$$

By Lemma 5 we have:

$$method_p(m, \overline{\theta}, \pi) = \{(m[\overline{\alpha} \triangleleft \overline{K}] (\overline{-} : \overline{\kappa}) : \omega' \_ , \overline{\theta}'_1 < : \overline{\theta}'_2)\}$$

where:

$$\begin{array}{c}
p; \Delta \vdash \omega' < : \omega \\
p; \Delta \vdash \overline{\theta}'_1 < : \overline{\theta}_2.
\end{array}$$

Noticing that:

$$\begin{aligned}
& path_{p\Delta\Gamma}(path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ as } (\sigma, \overline{\rho}, \overline{\pi}).m[\overline{\nu}](path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e'))) \\
&= path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ as } (\sigma, \overline{\rho}, \overline{\pi}).m[\overline{\nu}](path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e'))
\end{aligned}$$

and using [T-METHOD] finishes the case.

Case [T-TYPECASE]  $\frac{\begin{array}{c} p; \Delta; \Gamma \overline{x} : \overline{\tau} \vdash e : \pi \\ p; \Delta; \Gamma \overline{x} : \overline{\tau} \quad x : \sigma_i \vdash e_i : \nu_i \quad p; \Delta \vdash \nu_i < : \rho \quad 1 \leq i \leq |\overline{\sigma}| \\ p; \Delta; \Gamma \overline{x} : \overline{\tau} \quad x : \pi \vdash e' : \kappa \quad p; \Delta \vdash \kappa < : \rho \end{array}}{p; \Delta; \Gamma \overline{x} : \overline{\tau} \vdash \text{typecase } x = e \text{ in } \overline{\sigma} \Rightarrow \overline{e} \text{ else } e' \text{ end} : \rho}$ :

By the induction hypothesis we have:

$$\begin{array}{c}
p; \Delta; \Gamma \vdash path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) : \pi' \\
p; \Delta \vdash \pi' < : \pi \\
p; \Delta; \Gamma \quad x : \sigma_i \vdash path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e_i) : \nu'_i \\
p; \Delta \vdash \nu'_i < : \nu_i \\
p; \Delta; \Gamma \quad x : \pi \vdash path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e') : \kappa' \\
p; \Delta \vdash \kappa' < : \kappa.
\end{array}$$

By [S-TRANS] we have:

$$\begin{array}{c}
p; \Delta \vdash \nu'_i < : \rho \\
p; \Delta \vdash \kappa' < : \rho.
\end{array}$$

Noticing that:

$$\begin{aligned}
& path_{p\Delta\Gamma}(\text{typecase } x = path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ in } \overline{\sigma} \Rightarrow path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e) \text{ else } path_{p\Delta\Gamma}([\overline{e} / \overline{x}]e') \text{ end}) \\
&= path_{p\Delta\Gamma}(\text{typecase } x = [\overline{e} / \overline{x}]e \text{ in } \overline{\sigma} \Rightarrow [\overline{e} / \overline{x}]e \text{ else } [\overline{e} / \overline{x}]e' \text{ end})
\end{aligned}$$

and using [T-TYPECASE] finishes the case.  $\square$

**Lemma 7** (Type Substitution Preserves Typing). *If  $p$  is well typed and  $p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e : \tau$  and none of  $\overline{\alpha}$  appear in  $\Delta$  and  $p; \Delta \vdash \overline{\sigma} <: [\overline{\sigma}/\overline{\alpha}] \overline{K}$  where  $p; \Delta \vdash \overline{\sigma}$  ok then  $p; \Delta; [\overline{\sigma}/\overline{\alpha}] \Gamma \vdash \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}]e) : \rho$  and  $p; \Delta \vdash \rho <: [\overline{\sigma}/\overline{\alpha}] \tau$ .*

*Proof.* By induction on the typing derivation  $p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e : \tau$  with case analysis on the last rule applied.

Case [T-VAR], [T-SELF]: Trivial.

[T-OBJECT]

$$\begin{array}{c} \text{object } O[\overline{\alpha'} \triangleleft -] (\overline{-} : \overline{\pi}) - \in p \\ p; \Delta \overline{\alpha} <: \overline{K} \vdash O[\overline{\rho}] \text{ ok} \\ \text{Case } \frac{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash \overline{e} : \overline{v}}{p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{v} <: [\overline{\rho}/\overline{\alpha'}] \overline{\pi}} : \\ \frac{p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{v} <: [\overline{\rho}/\overline{\alpha'}] \overline{\pi}}{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash O[\overline{\rho}] (\overline{e}) : O[\overline{\rho}]} \end{array}$$

By Lemma 2, the induction hypothesis, and Lemma 4 we have:

$$\begin{array}{l} p; \Delta \vdash [\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}] \text{ ok} \\ p; \Delta; [\overline{\sigma}/\overline{\alpha}] \Gamma \vdash \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] \overline{e}) : \overline{v'} \\ p; \Delta \vdash \overline{v'} <: [\overline{\sigma}/\overline{\alpha}] \overline{v} \\ p; \Delta \vdash [\overline{\sigma}/\overline{\alpha}] \overline{v} <: [\overline{\sigma}/\overline{\alpha}] [\overline{\rho}/\overline{\alpha'}] \overline{\pi}. \end{array}$$

Notice that:

$$\begin{array}{l} \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}] (\text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] \overline{e}))) \\ = \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}] ([\overline{\sigma}/\overline{\alpha}] \overline{e})). \end{array}$$

Rules [S-TRANS] and [T-OBJECT] finish the case.

Case [T-ASCRPTION]: Similar to the previous case.

[T-FIELD]

$$\begin{array}{c} \text{object } O[\overline{\alpha'} \triangleleft -] (\overline{x} : \overline{\pi}) - \in p \\ \text{Case } \frac{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e : O[\overline{\rho}]}{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e.x_i : [\overline{\rho}/\overline{\alpha'}] \pi_i} : \end{array}$$

By induction hypothesis we have:

$$\begin{array}{l} p; \Delta; [\overline{\sigma}/\overline{\alpha}] \Gamma \vdash \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] e) : [\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}] \\ p; \Delta \vdash [\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}] <: [\overline{\sigma}/\overline{\alpha}] O[\overline{\rho}]. \end{array}$$

Notice that:

$$\begin{array}{l} \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}(\text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] e).x_i) \\ = \text{path}_{p\Delta[\overline{\sigma}/\overline{\alpha}]\Gamma}([\overline{\sigma}/\overline{\alpha}] e.x_i). \end{array}$$

Rule [T-FIELD] finishes the case.

[T-METHOD]

$$\begin{array}{c} p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e : \rho \quad p; \Delta \overline{\alpha} <: \overline{K} \vdash \rho \overline{\pi} \text{ path-ok} \\ p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{v} \text{ ok} \\ \text{Case } \frac{\text{method}_p(m, \rho \overline{\pi}, \overline{v}) = \{(m[\overline{\alpha'} \triangleleft K'](\overline{x} : \overline{\omega}) : \theta \_, \overline{\phi}_1 <: \overline{\phi}_2)\}}{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash \overline{e} : \overline{\omega'} \quad p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{\omega'} <: [\overline{\kappa}/\overline{\alpha'}] \overline{\omega}} : \\ \frac{p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{\kappa} \text{ ok} \quad p; \Delta \overline{\alpha} <: \overline{K} \vdash \overline{\kappa} <: [\overline{\kappa}/\overline{\alpha'}] K' \quad p; \Delta \vdash \overline{\phi}_1 <: \overline{\phi}_2}{p; \Delta \overline{\alpha} <: \overline{K}; \Gamma \vdash e \text{ as } (\rho \overline{\pi}, \overline{v}).m[\overline{\kappa}'](\overline{e}) : [\overline{\kappa}/\overline{\alpha'}] \theta} \end{array}$$

By the induction hypothesis we have:

$$\begin{aligned} p; \Delta; [\vec{\sigma}/\vec{\alpha}]\Gamma &\vdash \text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]e) : \rho' \\ p; \Delta &\vdash \rho' <: [\vec{\sigma}/\vec{\alpha}]\rho \\ p; \Delta; [\vec{\sigma}/\vec{\alpha}]\Gamma &\vdash \text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]\vec{e}) : \vec{\omega}'' \\ p; \Delta &\vdash \vec{\omega}'' <: [\vec{\sigma}/\vec{\alpha}]\vec{\omega}'. \end{aligned}$$

By Lemma 2 we have:

$$\begin{aligned} p; \Delta &\vdash [\vec{\sigma}/\vec{\alpha}]\vec{\nu} \text{ ok} \\ p; \Delta &\vdash [\vec{\sigma}/\vec{\alpha}]\vec{\kappa} \text{ ok}. \end{aligned}$$

By Lemma 4 we have:

$$\begin{aligned} p; \Delta &\vdash [\vec{\sigma}/\vec{\alpha}]\vec{\kappa}' <: [\vec{\sigma}/\vec{\alpha}][\vec{\kappa}/\vec{\alpha}']\vec{K}' \\ p; \Delta &\vdash [\vec{\sigma}/\vec{\alpha}]\vec{\omega}' <: [\vec{\sigma}/\vec{\alpha}][\vec{\kappa}/\vec{\alpha}']\vec{\omega}. \end{aligned}$$

By [S-TRANS] we have:

$$p; \Delta \vdash \vec{\omega}'' <: [\vec{\sigma}/\vec{\alpha}][\vec{\kappa}/\vec{\alpha}']\vec{\omega}.$$

Notice that:

$$\begin{aligned} &\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}(\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]e) \text{ as } [\vec{\sigma}/\vec{\alpha}](\rho\vec{\pi}, \vec{\nu}).m[[\vec{\sigma}/\vec{\alpha}]\vec{\kappa}'](\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]\vec{e}))) \\ &= \text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]e) \text{ as } (\vec{\pi}', [\vec{\sigma}/\vec{\alpha}]\vec{\nu}).m[[\vec{\sigma}/\vec{\alpha}]\vec{\kappa}'](\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]\vec{e})) \end{aligned}$$

where:

$$\text{min-path}_{p\Delta}(\rho', [\vec{\sigma}/\vec{\alpha}](\pi\vec{\nu})) = \vec{\pi}'.$$

By Lemma 5 we have:

$$\text{method}_p(m, \vec{\pi}', [\vec{\sigma}/\vec{\alpha}]\vec{\nu}) = \{(m[[\vec{\sigma}/\vec{\alpha}]\vec{\kappa}'](\vec{x}:[\vec{\sigma}/\vec{\alpha}]\vec{\omega}):[\vec{\sigma}/\vec{\alpha}]\theta' \_, \vec{\phi}'_1 <: \vec{\phi}'_2)\}$$

where:

$$\begin{aligned} p; \Delta &\vdash [\vec{\sigma}/\vec{\alpha}]\theta' <: [\vec{\sigma}/\vec{\alpha}]\theta \\ p; \Delta &\vdash \vec{\phi}'_1 <: \vec{\phi}'_2. \end{aligned}$$

Noticing that:

$$\begin{aligned} &\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}(\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]e) \text{ as } [\vec{\sigma}/\vec{\alpha}](\rho\vec{\pi}, \vec{\nu}).m[[\vec{\sigma}/\vec{\alpha}]\vec{\kappa}'](\text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]\vec{e}))) \\ &= \text{path}_{p\Delta[\vec{\sigma}/\vec{\alpha}]\Gamma}([\vec{\sigma}/\vec{\alpha}]e) \text{ as } [\vec{\sigma}/\vec{\alpha}](\rho\vec{\pi}, \vec{\nu}).m[[\vec{\sigma}/\vec{\alpha}]\vec{\kappa}']([\vec{\sigma}/\vec{\alpha}]\vec{e}) \end{aligned}$$

and using [T-METHOD] finishes the case.

Case [T-TYPECASE]: Similar to the previous case. □

**Lemma 8.** *If  $p; \Delta; \Gamma \vdash e : \tau$  then  $\text{path}_{p\Delta\Gamma}(e) = e$ .*

*Proof.* By induction on the structure of  $e$ . □

**Lemma 9 (Replacement).** *If  $p; \Delta; \Gamma \vdash E[e_1] : \tau_1$  with the subderivation  $p; \Delta; \Gamma \vdash e_1 : \sigma_1$  and  $p; \Delta; \Gamma \vdash e_2 : \sigma_2$  where  $p; \Delta \vdash \sigma_2 <: \sigma_1$  then  $p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e_2]) : \tau_2$  where  $p; \Delta \vdash \tau_2 <: \tau_1$ .*

*Proof.* This proof is by induction on the structure of  $E$ .

Case □: Then  $\tau_1 = \sigma_1$  and  $\tau_2 = \sigma_2$ . Lemma 8 gives the final result.

Case  $O[\vec{\tau}](\vec{v}E'\vec{e})$ : By the well-typedness of  $E[e_1]$  we have:

$$p; \Delta; \Gamma \vdash E[e_1] : \rho.$$

By [T-OBJECT] we have:

$$p; \Delta \vdash \rho <: [\vec{\tau}/\vec{\alpha}]\pi_i$$

where object  $O[\overline{\alpha \triangleleft -}](\overline{-}:\overline{\pi})_- \in p$ . By the induction hypothesis we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e'_2]) : \rho'$$

where:

$$p; \Delta \vdash \rho' <: \rho.$$

By [S-TRANS] we have:

$$p; \Delta \vdash \rho' <: [\overline{\tau} / \overline{\alpha}] \pi_i.$$

Notice that:

$$\text{path}_{p\Delta\Gamma}(O[\overline{\tau}](\overline{v} E[e'_2] \overline{e})) = O[\overline{\tau}](\overline{v} \text{path}_{p\Delta\Gamma}(E[e'_2]) \overline{e}).$$

By [T-OBJECT] we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e'_2]) : \tau_1.$$

Notice that the lemma holds with  $\tau_1 = \tau_2$ .

Case  $v$  as  $(\overline{\tau}, \overline{\sigma}).m[\overline{\rho}](\overline{v} E' \overline{e})$ : Similar to the previous case.

Case  $E'.x$ : By [T-FIELD] we have:

$$p; \Delta; \Gamma \vdash E[e'_1] : O[\overline{\tau}]$$

By the induction hypothesis we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e'_2]) : \rho$$

where:

$$p; \Delta \vdash \rho <: O[\overline{\tau}]$$

Therefore  $\rho = O[\overline{\tau}]$ . Notice that:

$$\text{path}_{p\Delta\Gamma}(E[e'_2].x) = \text{path}_{p\Delta\Gamma}(E[e'_2]).x$$

By [T-FIELD] we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e'_2]) : \tau_1.$$

Notice that the lemma holds with  $\tau_1 = \tau_2$ .

Case  $E'$  as  $(\tau \overline{\sigma}, \overline{\rho}).m[\overline{\pi}](\overline{e})$ : By [T-METHOD] we have:

$$p; \Delta; \Gamma \vdash E[e'_1] : \tau.$$

By the induction hypothesis we have:

$$p; \Delta; \Gamma \vdash \text{path}_{p\Delta\Gamma}(E[e'_2]) : \tau'$$

where:

$$p; \Delta \vdash \tau' <: \tau.$$

Notice that:

$$\begin{aligned} \text{path}_{p\Delta\Gamma}(E[e'_2]) \text{ as } (\tau \overline{\sigma}, \overline{\rho}).m[\overline{\pi}](\overline{e}) \\ = \text{path}_{p\Delta\Gamma}(E[e'_2]) \text{ as } (\overline{\sigma}', \overline{\rho}).m[\overline{\pi}](\overline{e}). \end{aligned}$$

where

$$\text{min-path}_{p\Delta}(\tau', \tau \overline{\sigma}) = \overline{\sigma}'.$$

By Lemma 5, if:

$$\text{method}_p(m, \tau \overline{\sigma}, \overline{\rho}) = \{(m[\overline{\alpha \triangleleft K}](\overline{x}:\overline{v}):\kappa_-, -)\}$$

then:

$$\text{method}_p(m, \overline{\sigma}', \overline{\rho}) = \{(m[\overline{\alpha \triangleleft K}](\overline{x}:\overline{v}):\kappa'_, -)\}$$

where:

$$p; \Delta \vdash \kappa' <: \kappa.$$

Rule [T-METHOD] finishes the case.

Case typecase  $x = E'$  in  $\overrightarrow{\tau} \Rightarrow \vec{e}$  else  $e$  end: Similar to the previous case.  $\square$

**Theorem 2** (Subject Reduction). *If  $p$  is well typed and  $p; \emptyset; \emptyset \vdash e_1 : \tau_1$  and  $p \vdash e_1 \longrightarrow e_2$  then  $p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}(e_2) : \tau_2$  where  $p; \emptyset \vdash \tau_2 <: \tau_1$ .*

*Proof.* The proof is by case analysis on the evaluation rule applied.

Case [R-FIELD]:  $e_1 = E[O[\overrightarrow{\tau}](\overrightarrow{v}).x_i]$   $e_2 = E[v_i]$

Then object  $O[\overrightarrow{\alpha} \triangleleft \overrightarrow{\sigma}](\overrightarrow{x}; \vec{\sigma}) \_ \in p$ . By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash O[\overrightarrow{\tau}](\overrightarrow{v}).x_i : [\overrightarrow{\tau} / \overrightarrow{\alpha}] \sigma_i.$$

By [T-OBJECT] we have:

$$p; \emptyset; \emptyset \vdash v_i : \rho_i$$

where:

$$p; \emptyset \vdash \rho_i <: [\overrightarrow{\tau} / \overrightarrow{\alpha}] \sigma_i.$$

By Lemma 9:

$$p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}(E[v_i]) : \tau_2$$

where:

$$p; \emptyset \vdash \tau_2 <: \tau_1.$$

Case [R-METHOD]:  $e_1 = E[O[\overrightarrow{\tau}](\overrightarrow{v}) \text{ as } (\overrightarrow{\sigma}, \overrightarrow{\rho}).m[\overrightarrow{\pi}](\overrightarrow{v}')] ]$   $e_2 = E[\overrightarrow{v} / \overrightarrow{x}][O[\overrightarrow{\tau}](\overrightarrow{v}) / \text{self}][\overrightarrow{v}' / \overrightarrow{x}'][\pi / \alpha'] e]$

Then:

$$\begin{aligned} \text{object } O[\overrightarrow{\alpha} \triangleleft \overrightarrow{\sigma}](\overrightarrow{x}; \vec{\sigma}) \_ \in p \\ \text{method}_p(m, \overrightarrow{\sigma}, \overrightarrow{\rho}) = \{(m[\overrightarrow{\alpha}' \triangleleft K'])(\overrightarrow{x}'; \vec{\kappa}) : \omega = e, \_ \} \end{aligned}$$

By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash O[\overrightarrow{\tau}](\overrightarrow{v}) \text{ as } (\overrightarrow{\sigma}, \overrightarrow{\rho}).m[\overrightarrow{\pi}](\overrightarrow{v}') : [\overrightarrow{\pi} / \overrightarrow{\alpha}] \omega.$$

There are two subcases to consider.

Subcase:  $m$  is defined in  $O$

By [T-METHODDEF] and Lemma 7 we have:

$$p; \overrightarrow{\alpha}' <: \overrightarrow{K}'; \text{self} : O[\overrightarrow{\tau}](x : \overrightarrow{v}' \ x' : \vec{\kappa}) \vdash \text{path}_{p\Delta\Gamma}(e) : \omega'$$

where:

$$\begin{aligned} \overrightarrow{v}' &= [\overrightarrow{\tau} / \overrightarrow{\alpha}] \overrightarrow{v} \\ p; \overrightarrow{\alpha}' <: \overrightarrow{K}' \vdash \omega' <: \omega. \end{aligned}$$

By [T-METHOD] and Lemma 4 we have:

$$\begin{aligned} p; \emptyset \vdash \overrightarrow{\pi} \text{ ok} \\ p; \emptyset \vdash \overrightarrow{\pi} <: [\overrightarrow{\pi} / \overrightarrow{\alpha}'] \overrightarrow{K}' \\ p; \emptyset; \emptyset \vdash \overrightarrow{v}' : \vec{\kappa}' \\ p; \emptyset \vdash \vec{\kappa}' <: [\overrightarrow{\pi} / \overrightarrow{\alpha}'] \vec{\kappa}. \end{aligned}$$

By [T-OBJECT] and Lemma 4 we have:

$$\begin{aligned} p; \emptyset; \emptyset \vdash \overrightarrow{v} : \overrightarrow{v}'' \\ p; \emptyset \vdash \overrightarrow{v}'' <: [\overrightarrow{\pi} / \overrightarrow{\alpha}'] \overrightarrow{v}'. \end{aligned}$$

By Lemma 6, Lemma 7, and [S-TRANS] we have:

$$\begin{aligned} p; \emptyset; \emptyset \vdash [\overrightarrow{v} / \overrightarrow{x}][O[\overrightarrow{\tau}](\overrightarrow{v}) / \text{self}][\overrightarrow{v}' / \overrightarrow{x}'][\pi / \alpha'] \text{path}_{p\emptyset\emptyset}(e) : \omega'' \\ p; \emptyset \vdash \omega'' <: [\overrightarrow{\pi} / \overrightarrow{\alpha}'] \omega'. \end{aligned}$$

By Lemma 9:

$$p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}(E[\vec{v}/\vec{x}][O[\vec{\tau}](\vec{v})/\mathbf{self}][\vec{v}'/\vec{x}'][\pi/\alpha']\text{path}_{p\emptyset\emptyset}(e)]) : \tau_2$$

where:

$$p; \emptyset \vdash \tau_2 <: \tau_1.$$

Noticing that:

$$\begin{aligned} & \text{path}_{p\emptyset\emptyset}(E[\vec{v}/\vec{x}][O[\vec{\tau}](\vec{v})/\mathbf{self}][\vec{v}'/\vec{x}'][\pi/\alpha']\text{path}_{p\emptyset\emptyset}(e)]) \\ &= \text{path}_{p\emptyset\emptyset}(E[\vec{v}/\vec{x}][O[\vec{\tau}](\vec{v})/\mathbf{self}][\vec{v}'/\vec{x}'][\pi/\alpha']e]) \end{aligned}$$

finishes the case.

Subcase:  $m$  is not defined in  $O$

Almost identical to the above case. The only difference is that substituting the object's value parameters is superfluous.

Case [R-TYPECASE]:

$$e_1 = E[\text{typecase } x = O[\vec{\tau}](\vec{v}) \text{ in } \overrightarrow{\sigma} \Rightarrow e'_1 \rho \Rightarrow e'_2 \pi \Rightarrow e'_3 \text{ else } e'_4 \text{ end}]$$

$$e_2 = E[O[\vec{\tau}](\vec{v})/x]e'_2]$$

Then:

$$p; \emptyset \vdash O[\vec{\tau}] <: \rho.$$

By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash \text{typecase } x = O[\vec{\tau}](\vec{v}) \text{ in } \overrightarrow{\sigma} \Rightarrow e'_1 \rho \Rightarrow e'_2 \pi \Rightarrow e'_3 \text{ else } e'_4 \text{ end} : \nu$$

where:

$$\begin{aligned} & p; \emptyset; x : \rho \vdash e'_2 : \nu' \\ & p; \emptyset \vdash \nu' <: \nu. \end{aligned}$$

By Lemma 6 we have:

$$p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}([O[\vec{\tau}](\vec{v})/x]e'_2) : \nu''$$

where:

$$p; \emptyset \vdash \nu'' <: \nu'.$$

By Lemma 9:

$$p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}(E[\text{path}_{p\emptyset\emptyset}([O[\vec{\tau}](\vec{v})/x]e'_2)]) : \tau_2$$

where:

$$p; \emptyset \vdash \tau_2 <: \tau_1.$$

Noticing that:

$$\begin{aligned} & \text{path}_{p\emptyset\emptyset}(E[\text{path}_{p\emptyset\emptyset}([O[\vec{\tau}](\vec{v})/x]e'_2)]) \\ &= \text{path}_{p\emptyset\emptyset}(E[O[\vec{\tau}](\vec{v})/x]e'_2]) \end{aligned}$$

finishes the case.

Case [R-TYPECASEELSE]: Similar to the previous case.

Case [R-ASCRPTION]:  $e_1 = E[e \text{ as } \tau]$   $e_2 = E[e]$

By the well-typedness of  $e_1$  we have:

$$p; \emptyset; \emptyset \vdash e \text{ as } \tau : \tau$$

where:

$$\begin{aligned} & p; \emptyset; \emptyset \vdash e : \sigma \\ & p; \emptyset \vdash \sigma <: \tau. \end{aligned}$$

By Lemma 9:

$$p; \emptyset; \emptyset \vdash \text{path}_{p\emptyset\emptyset}(E[e]) : \tau_2$$

where:

$$p; \emptyset \vdash \tau_2 <: \tau_1.$$

□

**Theorem 3** (Type Soundness). *If  $p; \emptyset; \emptyset \vdash e_1 : \tau_1$  then either  $e_1$  is a value or*

$$\begin{aligned} p \vdash e_1 &\longrightarrow e_2, \\ p \vdash \mathit{path}_{p\emptyset\emptyset}(e_2) &\longrightarrow e_3, \\ \dots & \\ p \vdash \mathit{path}_{p\emptyset\emptyset}(e_n) &\longrightarrow v. \end{aligned}$$

*Proof.* By induction on the structure of  $e_1$  using Theorems 1 and 2. □