# A Differential Inequalities method for Verified Solution of IVPs for ODEs using Convex or Linear Programming for the Search of Tight Bounds

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**Abstract**

Nedialkov stated that "Although high-order Taylor series may be reasonably efficient for mildly stiff ODEs, we do not have an interval method suitable for stiff ODEs." [1] This paper suggests such a method, based on differential inequalities. Assume we have an IVP for ODE $\dot{y}(t) = f(y(t), t)$, $y(0) \in [y_0]$, $y(t) \in \mathbb{R}^n$ and we want to compute the bounds of $y(h)$. Assume also that the right-hand side function $f(y, t)$ is bounded by lower and upper bounds $\underline{f}(y, t) \le f(y, t) \le \overline{f}(y, t)$. As in [2], we use Müller's theorem from the theory of differential inequalities. We need to guess $2n$ functions $\underline{y}_i(t)$ and $\overline{y}_i(t)$ such that

$$[y_{0i}] \subset [\underline{y}_i(0), \overline{y}_i(0)] \tag{1}$$

$$\dot{\underline{y}}_i(t) \le \underline{f}_i(y, t) \qquad \text{if } y \in [\underline{y}(t), \overline{y}(t)] \text{ and } y_i = \underline{y}_i(t) \tag{2}$$

$$\dot{\overline{y}}_i(t) \ge \overline{f}_i(y, t) \qquad \text{if } y \in [\underline{y}(t), \overline{y}(t)] \text{ and } y_i = \overline{y}_i(t) \tag{3}$$

for all $t \in [0, h]$. Then $y_i(h) \in [\underline{y}_i(h), \overline{y}_i(h)]$. The conditions (1)-(3) can be verified for given functions $\underline{y}(t)$ and $\overline{y}(t)$ by interval computation [3].

For a test IVP $\dot{y}(t) = \lambda y$, $y(0) \in [0, 1]$ ($\lambda < 0$) we guess a pair of linear functions $[\underline{y}(t), \overline{y}(t)] = [0, \frac{1+\lambda(t-h)}{1-\lambda h}]$ which satisfy (1)-(3). The bound of $y(h)$ is $[0, \frac{1}{1-\lambda h}]$. Its width decays to zero when $h \to \infty$, demonstrating suitability of the method for stiff ODEs. However, the problem is to find the bound functions in a regular way.

If we restrict $\underline{y}$ and $\overline{y}$ by polynomials of degree $r$, the search space will have dimension $2n(r+1)$. Consider a few optimization problems in this search space with common constraints (1)-(3) and with targets like $\underline{y}_i(h) \to max$, $\overline{y}_i(h) \to min$, $\overline{y}_i(h) - \underline{y}_i(h) \to min$. Each problem returns a bounding box for $y(h)$. The intersection of these boxes also bounds $y(h)$.

If $\underline{f}(y,t)$ is concave function of $y$ and $\overline{f}(y,t)$ is convex function of $y$ for any $t \in [0,h]$, then each optimization problem becomes a convex problem. A local minimum of a convex problem is a global minimum [4]. Though the bounds maybe high-order polynomials, the optimization algorithm requires only low-order derivatives of $f(y,t)$. This is good for applicatons with an empirical region-wise definition $f(y,t)$.

Differential inequalities can be used also to bound the deviation of the true solution from the approximate solution $\hat{y}$ found by other validated or non-validated methods. In this case we can bound the function by its linear approximation near the approximate solution $f(y,t) \in f^I(\hat{y}(t),t) + [J(t)](y - \hat{y}(t))$. When $[f(y,t)] = [J]y + [b(t)]$, $\underline{b}(t)$ and $\overline{b}(t)$ are linear and $r = 1$, the optimization problem becomes a linear programming problem. The initial solution basis for the LP algorithm at the timestep can be borrowed from the previous timestep to have fewer iterations.

**References:**


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